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Stability of a Dual-Spin Satellite in Circular Orbit

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Nomenclature

a = distance from pendulum damper mass to 0
A_0 = moment of inertia of undeformed configuration about $0'x'$ or $0'y'$
$A(\tau),B(\tau),C(\tau),D(\tau)$ = variables of motion defined in Eqs. (8)
a_{11},a_{21} = constants defined in Eqs. (12)
b_1,b_2 = damping constants of platform and rotor dampers, respectively
c_1,c_2 = dimensionless damping constants, $b_1/m\Omega$ and $b_2/I_s\Omega$, respectively
d_{11},d_{21} = constants defined in Eqs. (13)
C_0 = moment of inertia of undeformed configuration about $0'z'$
C_R = moment of inertia of rotor (including sphere) about $0'z'$
G_1,G_2 = spring constants of platform and rotor dampers, respectively
I_s = moment of inertia of rotor damper about its center of mass
I = dimensionless inertia, I_s/A_0
J = C_R/A_0
K_1,K_2 = constants defined in Eq. (6)
k_1,k_2 = dimensionless spring constants $(G_1/m\Omega^2) - 3(1-\mu)$ and $G_2/I_s\Omega^2$, respectively
m = mass of platform damper
0 = instantaneous mass center of configuration
$0'$ = mass center of undeformed configuration
$0'x'y'z'$ = reference frame associated with undeformed configuration, with $0'x'$ and $0'y'$ aligned along principal axes of the platform, and $0'z'$ aligned along the common principal axes of the two bodies
$0xyz$ = reference frame associated with deformed configuration, constrained to be parallel to $0'x'y'z'$
p,p_1,p_2 = natural frequencies
R = dimensionless inertia, ma^2/A_0
t = time
W_1,W_2 = dimensionless variables, ω_x/Ω and ω_y/Ω, respectively
α = dimensionless rotation rate, $\dot\gamma_0/\Omega$
β = angular deflection of rotor damper
γ = rotation angle of rotor with respect to platform
$\dot\gamma_0$ = angular speed of rotor with respect to platform
Δ = $(C_0-A_0)/A_0$
ϵ = small parameter
ξ = dimensionless displacement, χ/a
Ξ = determinant defined after Eq. (10)
Λ_1,Λ_2 = expressions defined after Eqs. (4)
τ = dimensionless time, Ωt
μ = ratio of m to satellite mass
Ω = angular rate of circular orbit
$\omega_x,\omega_y,\omega_z$ = component absolute angular velocities of $0xyz$ reference frame
ψ,θ,ϕ = Euler orientation angles, depicted in Fig. 1
χ = linear deflection of platform damper
$(\dot{\ })$ = denotes differiation with respect to time
$(\ )'$ = denotes differentiation with respect to dimensionless time, τ
$\langle\ \rangle$ = denote "averaged" variables

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Introduction

RECENTLY, a "method of averaging" formulation has been used to solve linearized equations describing the motion of a dual-spin satellite rotating in a force-free environment.[1] In the present Note, the formulation and results are extended to account for the influence of the Earth's gravitational field when the satellite is in circular orbit with spin axis approximately normal to the orbit plane.

Equations of Motion

Paralleling Ref. 1, the dual spin configuration chosen for study is composed of a platform which contains a pendulum damper and a rotor which contains a spherical damper, as in Fig. 1. The rotor rotates with respect to the platform about $0'z'$ with angle γ. The satellite axes ($0xyz$) are referenced to orbital axes ($0A_1A_2A_3$) by a set of Euler angles (ψ,θ,ϕ) generated by the right-hand rotation scheme in Fig. 2: 1) ψ about $0A_1$, leading to axes ($0B_1B_2B_3$); 2) θ about $0B_2$ leading to axes ($0C_1C_2C_3$); 3) ϕ about $0C_3$ leading to axes $0xyz$. Linearized kinematic equations relating the spacecraft angular rates to the Euler angles are

$$\dot\theta = \omega_x - \Omega\psi;\quad \dot\psi = \omega_y + \Omega\theta;\quad \dot\phi = \omega_z - \Omega \tag{1}$$

Both rotor and platform are assumed to be symmetric about $0'z'$ when the configuration is undeformed, and consequently the linearized expression for gravitational torque about axes $0z$ is identically zero. We assume that the rotation angle γ is controlled by supplying a torque with an internal motor in the manner outlined in Ref. 1, which results in the rotation rate being a constant $\dot\gamma_0$ and ϕ being zero, under steady-state conditions. Consequently Eqs. (4, 5, and 8) of Ref. 1 are preserved.

The equilibrium solution of interest corresponds to pure rotation of the satellite about the $0A_3$ axis (and hence about $0z$) with the platform pointing at the Earth, i.e.,

$$\phi = 0;\quad \gamma = \dot\gamma_0 t;\quad \omega_z = \Omega \tag{2a}$$

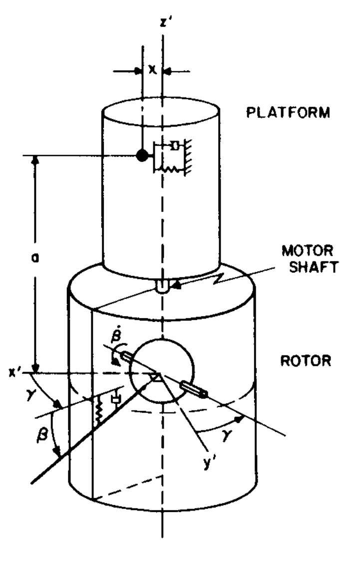

Fig. 1 Dual-spin satellite configuration.

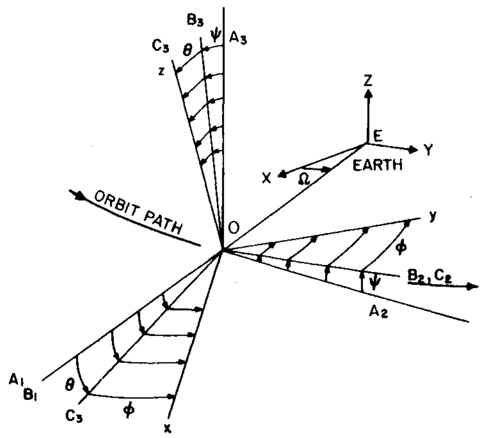

Fig. 2 Rotation scheme defining Euler angles.

$$\psi = \theta = \omega_x = \omega_y = \chi = \beta = 0 \quad (2b)$$

The solution (2a) satisfies identically the third statement in Eq. (1) together with Eqs. (4) and (5) of Ref. 1. Four dynamical motion equations linearized about solution (2b), readily derived by paralleling the development of Eqs. (9–12) of Ref. 1 (with minor additions to account for gravitational forces) and subsequent use of the first and second statements of Eq. (1), are

$$\psi'' + (\Delta + J\alpha)\psi + (\Delta + J\alpha - 1)\theta' = \epsilon\Lambda_1 \quad (3a)$$

$$-(\Delta + J\alpha - 1)\psi' + \theta'' + (4\Delta + J\alpha)\theta = \epsilon\Lambda_2 \quad (3b)$$

$$(1 - \mu)\xi'' + c_1\xi' + k_1\xi = -\theta'' - 2\psi' + 4\theta \quad (4a)$$

$$\beta'' + c_2\beta' + k_2\beta = \alpha(\psi' - \theta)\cos\gamma + (\psi'' - \theta')\sin\gamma + \alpha(\theta' + \psi)\sin\gamma - (\theta'' + \psi')\cos\gamma \quad (4b)$$

where

$$\Lambda_1 = \epsilon^{-1}\{2R\xi' + I\beta''\sin\gamma + I(\alpha + 1)\beta'\cos\gamma\}$$

$$\Lambda_2 = \epsilon^{-1}\{R(-\xi'' + 4\xi) - I\beta''\cos\gamma + I(\alpha + 1)\beta'\sin\gamma\}$$

$\tau = \Omega t$, and the primes denote differentiation with respect to τ, and $\alpha = \dot{\gamma}_0/\Omega$. The dimensionless constants α, c_1, k_1, etc, are as defined in Ref. 1, but are normalized with respect to a particular ω_z equal to Ω, in this instance.

Solutions When the Dampers are Absent

With the dampers absent, R and I are zero, consequently Λ_1 and Λ_2 are zero, and Eqs. (3) admit a solution

$$\psi = \delta_1\cos p_1\tau + \delta_2\sin p_1\tau + \delta_3\cos p_2\tau + \delta_4\sin p_2\tau \quad (5a)$$

$$\theta = K_1\delta_1\sin p_1\tau - K_1\delta_2\cos p_1\tau + K_2\delta_3\sin p_2\tau - K_2\delta_4\cos p_2\tau \quad (5b)$$

where δ_1, δ_2, δ_3, and δ_4 are arbitrary constants, the K's are given by

$$K_i = -(-p_i^2 + \Delta + J\alpha)/\{p_i(\Delta + J\alpha - 1)\} = -p_i(\Delta + J\alpha - 1)/(-p_i^2 + 4\Delta + J\alpha) \qquad i = 1,2 \quad (6)$$

and p_1^2 and p_2^2 are roots of

$$p^4 - \{(\Delta + J\alpha)^2 + 3\Delta + 1\}p^2 + (\Delta + J\alpha)(4\Delta + J\alpha) = 0 \quad (7)$$

Only two of four possible solutions of Eq. (7) are implied in Eq. (5); the positive branches are selected (without loss of generality) for the following work.

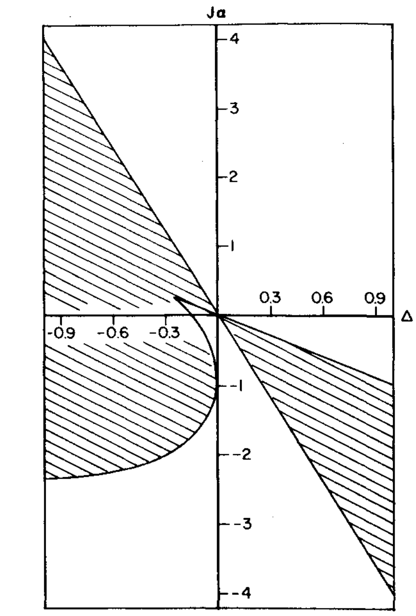

Fig. 3 Parameter plane for satellite stability when damping is absent (crosshatch denotes instability).

Stability, determined by the sign of the roots of Eq. (7), is summarized in chart form in Fig. 3. The chart is equivalent to the previously published Fig. 3 of Ref. 2, but has the added feature of portraying all regions of instability within a finite region, namely $|J\alpha| \leqslant 4$.

Transformation of Equations

New variables $A(\tau)$, $B(\tau)$, $C(\tau)$, and $D(\tau)$ are introduced to replace θ, ψ, θ', and ψ', via the transformation

$$\psi = A\cos p_1\tau + B\sin p_1\tau + C\cos p_2\tau + D\sin p_2\tau \quad (8a)$$

$$\psi' = -Ap_1\sin p_1\tau + Bp_1\cos p_1\tau - Cp_2\sin p_2\tau + Dp_2\cos p_2\tau \quad (8b)$$

$$\theta = K_1A\sin p_1\tau - K_1B\cos p_1\tau + K_2C\sin p_2\tau - K_2D\cos p_2\tau \quad (8c)$$

$$\theta' = K_1Ap_1\cos p_1\tau + K_1Bp_1\sin p_1\tau + K_2p_2C\cos p_2\tau + K_2p_2D\sin p_2\tau \quad (8d)$$

motivated by Eq. (5). Differentiation of Eq. (8a) and use of (8b) results in the additional constraint equation,

$$A'\cos p_1\tau + B'\sin p_1\tau + C'\cos p_2\tau + D'\sin p_2\tau = 0 \quad (9a)$$

Similarly Eqs. (8c) and (8d) yield,

$$K_1A'\sin p_1\tau - K_1B'\cos p_1\tau + K_2C'\sin p_2\tau - K_2D'\cos p_2\tau = 0 \quad (9b)$$

Substitution of Eqs. (8) into Eqs. (3) and further combination with Eqs. (9), results in,

$$A' = (\epsilon/\Xi)\{(p_2K_2^2 - p_1K_1K_2)\Lambda_1\sin p_1\tau + (p_1K_2 - p_2K_1)\Lambda_2\cos p_1\tau\} \quad (10a)$$

$$B' - (\epsilon/\Xi)\{-(p_2K_2^2 - p_1K_1K_2)\Lambda_1\cos p_1\tau + (p_1K_2 - p_2K_1)\Lambda_2\sin p_1\tau\} \quad (10b)$$

$$C' = (\epsilon/\Xi)\{(p_1K_1^2 - p_2K_1K_2)\Lambda_1\sin p_2\tau - (p_1K_2 - p_2K_1)\Lambda_2\cos p_2\tau\} \quad (10c)$$

$$D' = (\epsilon/\Xi)\{-(p_1K_1^2 - p_2K_1K_2)\Lambda_1\cos p_2\tau - (p_1K_2 - p_2K_1)\Lambda_2\sin p_2\tau\} \quad (10d)$$

where

$$\Xi = (p_1K_1 - p_2K_2)(p_1K_2 - p_2K_1)$$

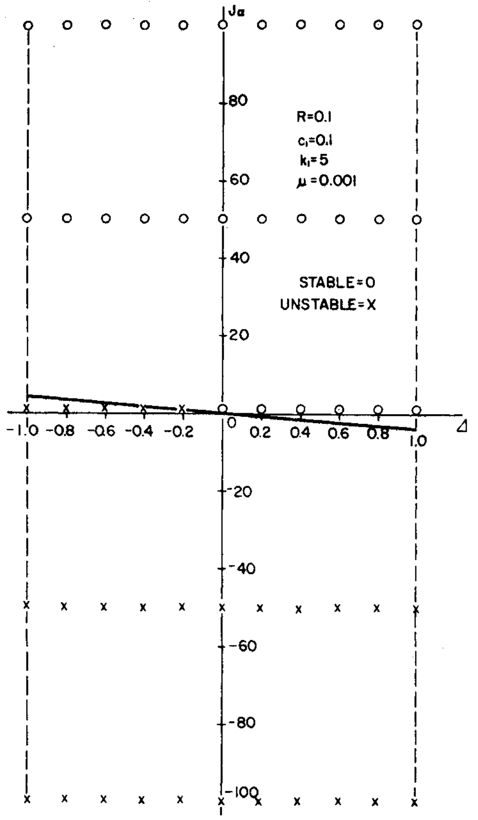

Fig. 4 Parameter plane demonstrating instability when $\alpha < 0$.

Combining Eqs. (8) with (4) yields,

$$(1-\mu)\xi'' + c_1\xi' + k_1\xi = (K_1p_1^2 + 2p_1 + 4K_1)(-A\sin p_1\tau + B\cos p_1\tau) + (K_2p_2^2 + 2p_2 + 4K_2)(C\sin p_2\tau - D\cos p_2\tau) - \epsilon\Lambda_1 \quad (11a)$$

$$\beta'' + c_2\beta' + k_2\beta = \{\alpha(p_1 + K_1) - p_1(K_1p_1 + 1)\}\{-A\sin p_1\tau + B\cos p_1\tau\}\cos\alpha\tau + \{\alpha(K_1p_1 + 1) - p_1(p_1 + K_1)\}\{A\cos p_1\tau + B\sin p_1\tau\}\sin\alpha\tau + \{\alpha(p_2 + K_2) - p_2(K_2p_2 + 1)\}\{-C\sin p_2\tau + D\cos p_2\tau\}\cos\alpha\tau + \{\alpha(K_2p_2 + 1) - p_2(p_2 + K_2)\}\{C\cos p_2\tau + D\sin p_2\tau\}\sin\alpha\tau + \epsilon(\Lambda_1\sin\gamma - \Lambda_2\cos\gamma) \quad (11b)$$

Equations (10) and (11) are exact in that they are derived from Eqs. (3) and (4) with no approximations. Furthermore, they are of a form amenable to solution by the method of averaging.[3] Although a first approximation solution is straightforward in principle, in reality it requires a great deal of effort. To make the algebra tractable a "high spin" approximation, which implies that $J\alpha$ is large, is introduced.

First Approximation for the High Spin Case

The roots of Eq. (7) may be obtained from the expansion

$$p^2 = (J\alpha)^2/2\ [\{1 + 2\Delta/J\alpha + (\Delta^2 + 3\Delta + 1)/(J\alpha)^2\} - \{1 + 2\Delta/J\alpha + (\Delta^2 + 3\Delta - 1)/(J\alpha)^2\} + 0\{1/(J\alpha)^3\}]$$

When $J\alpha$ is large, $p_1^2 \simeq 1$, $p_2^2 \simeq J^2\alpha^2$ and consequently $K_1 \simeq -1$, $K_2 \simeq 1$, and $\Xi \simeq -J^2\alpha^2$. Equations (10) and (11) then reduce substantially in complexity. Subsequent application of the method of averaging as outlined and demonstrated in Ref. 1 then results in the "averaged" set of differential equations,

$$\langle A\rangle' = a_{11}\langle A\rangle - a_{21}\langle B\rangle;\ \langle B\rangle' = a_{21}\langle A\rangle + a_{11}\langle B\rangle \quad (12)$$

$$\langle C\rangle' = d_{11}\langle C\rangle - d_{21}\langle D\rangle;\ \langle D\rangle' = d_{21}\langle C\rangle + d_{11}\langle D\rangle \quad (13)$$

where

$$a_{11} = -9c_1R/[2J\alpha\{(k_1-1)^2 + c_1^2\}]$$

$$a_{21} = 3R(k_1-1)/[2J\alpha\{(k_1-1)^2 + c_1\}^2]$$

$$d_{11} = -\frac{c_1R(J\alpha)^4}{2[(k_1 - J^2\alpha^2)^2 + (c_1J\alpha)^2]} - \frac{c_2I\alpha^2(J-1)^3(J\alpha)^2}{2J[\{k_2 - \alpha^2(J-1)^2\}^2 + \{c_2\alpha(J-1)\}^2]}$$

$$d_{21} = \frac{R(J\alpha)^2(k_1 - J^2\alpha^2)}{2J\alpha[(k_1 - J^2\alpha^2)^2 + (c_1J\alpha)^2]} + \frac{I\alpha(J-1)^2(J\alpha)^2\{k_2 - \alpha^2(J-1)^2\}}{2J[\{k_2 - \alpha^2(J-1)^2\}^2 + \{c_2\alpha(J-1)\}^2]}$$

and the $\langle\ \rangle$ denotes averaged value.

Discussion

Solutions of Eqs. (12) and (13) are stable in the usual mathematical sense (i.e., the motion is bounded when $t \to \infty$) when $a_{11} \leqslant 0$ and $d_{11} \leqslant 0$. The sign of a_{11} depends directly on the sign of α, and therefore the criteria for stability may be expressed as

$$\alpha \leqslant 0 \quad (14a)$$

$$d_{11} \leqslant 0 \quad (14b)$$

Comparable analytical results derived for a spacecraft in a force-free environment [i.e., Eqs. (21) of Ref. 1 with Q large] are found to be embodied exactly in the above Eqs. (13) and criterion (14b). The appearance of Eqs. (12) and the criterion (14a) thus stem from the inclusion of gravitational forces in the problem formulation.

Criterion (14b) requires no further comment in view of the large body of literature concerned with the freely-spinning dual spin satellite.[1] To provide a check on criterion (14a) Routh's Rules were applied to Eqs. (3) and (4) with $I = 0$ (thus the equations under consideration were (3a), (3b), and (4a), a set of linear equations with constant coefficients). The results, presented in Fig. 4, are seen to be in complete agreement with the criterion when $|\alpha|$ is greater than 10.

For satellite parameters which are typical of TACSAT or Intelsat IV[4] and spin direction such that α is negative, the previous theory indicates that a wobble with initial amplitude of 0.01 degrees would grow to 0.1 degrees in 1.6×10^{13} yr. Thus, the motion is technically unstable in that it is unbounded as $t \to \infty$, but the instability is of little consequence in current engineering practice.

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